

Universal Optimal Quantum Correlator

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Recently, a novel operational strategy to access quantum correlation functions of the form $\text{Tr}[A\rho B]$ was provided in [F. Buscemi, M. Dall'Arno, M. Ozawa, and V. Vedral, arXiv:1312.4240]. Here we propose a realization scheme, that we call *partial expectation values*, implementing such strategy in terms of a unitary interaction with an ancillary system followed by the measurement of an observable on the ancilla. Our scheme is *universal*, being independent of ρ , A , and B , and it is optimal in a statistical sense. Our scheme is suitable for implementation with present quantum optical technology, and provides a new way to test uncertainty relations.

Keywords: quantum correlation functions; ideal quantum correlator; partial expectation values

1. Introduction

Stochastic processes play a fundamental role in a plethora of fields such as classical and quantum statistics ¹, thermodynamics ², and field theory ³. They are successfully described in terms of correlation functions, namely expectation values of the product of dynamical variables. In classical theory, dynamical variables are represented by real functions, while in quantum theory they are represented by quantum observables – i.e. Hermitian operators. Both theories provide a prescription to di-

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rectly measure the expectation value of any single dynamical variable.

Classically, this prescription is sufficient to measure any correlation function, since products of dynamical variables are dynamical variables themselves. This is not the case in quantum theory, where the product of non-commuting observables is not an observable in general. Thus, while formally well-defined, quantum correlation functions appear to lack of a direct operational interpretation.

Recently ⁴, the present authors proposed a novel scheme – referred to as *ideal quantum correlator* – which allows to operationally access the expectation value of the product of any two observables A and B over any quantum state ρ , namely any two-point quantum correlation function $\text{Tr}[A\rho B]$. The scheme consists in a quantum preprocessing and classical postprocessing strategy which is *universal*, being independent of ρ, A, B , and is *optimal* in a statistical sense.

The aim of this work is to provide a simple realization scheme for our universal optimal strategy, in terms of a unitary interaction U with an ancillary system followed by the measurement of an observable Z on the ancilla. Our scheme is universal, U and Z being fixed and independent of ρ, A , and B , and optimal, minimizing the statistical error associated with the classical postprocessing. Our scheme is suitable for implementation with present quantum optical technology, and provides a new way to test uncertainty relations ⁵.

2. Formalization

Let us first fix the notation ⁶. Let \mathcal{H} and \mathcal{K} be some Hilbert spaces. We denote by $\mathbf{L}(\mathcal{H}, \mathcal{K})$ the set of all linear operators mapping elements in \mathcal{H} to elements in \mathcal{K} , with the convention that $\mathbf{L}(\mathcal{H}) := \mathbf{L}(\mathcal{H}, \mathcal{H})$. We denote by $\mathbf{S}(\mathcal{H})$ the set of all states, namely all those operators $\rho \in \mathbf{L}(\mathcal{H})$ such that $\rho \geq 0$ and $\text{Tr}[\rho] = 1$. The identity matrix is denoted by the symbol $\mathbb{1}$. Physical transformation mapping quantum states on \mathcal{H} to quantum states on \mathcal{K} are described by trace-preserving, completely positive linear maps $\mathcal{M} : \mathbf{L}(\mathcal{H}) \rightarrow \mathbf{L}(\mathcal{K})$.

3. Ideal Quantum Correlator

Formally, for any Hilbert space \mathcal{H} , the *ideal quantum correlator* is defined ⁴ as the map $\mathcal{T} : \mathbf{L}(\mathcal{H}) \rightarrow \mathbf{L}(\mathcal{H}) \otimes \mathbf{L}(\mathcal{H})$ such that

$$\text{Tr}[\mathcal{T}(\rho) (A \otimes B)] = \text{Tr}[A\rho B], \quad (1)$$

for any observables $A, B \in \mathbf{L}(\mathcal{H})$ and any state $\rho \in \mathbf{S}(\mathcal{H})$.

The ideal quantum correlator \mathcal{T} is not a physical map, as it is not Hermiticity preserving (HP). However, in Ref. ⁴ it was proved that its expectation value can be decomposed in terms of physical – namely, completely positive (CP) and trace not-increasing – maps. More precisely, it was shown that any HP linear map $\mathcal{L} : \mathbf{L}(\mathcal{H}) \rightarrow \mathbf{L}(\mathcal{K})$ can be decomposed as $\mathcal{L} = \sum_i \lambda_i \mathcal{E}_i$, where λ_i are real coefficients and $\mathcal{E}_i : \mathbf{L}(\mathcal{H}) \rightarrow \mathbf{L}(\mathcal{K})$ are completely positive linear maps whose average, $\mathcal{E} := \sum_i \mathcal{E}_i$, is trace-preserving. These are called “statistical decompositions” of map \mathcal{L} and are

illustrated in Fig. 1. In Ref. ⁴, the optimal decomposition of map \mathcal{T} minimizing the statistical error associated with the postprocessing was derived. In the next Section we will provide a realization scheme for such decomposition.

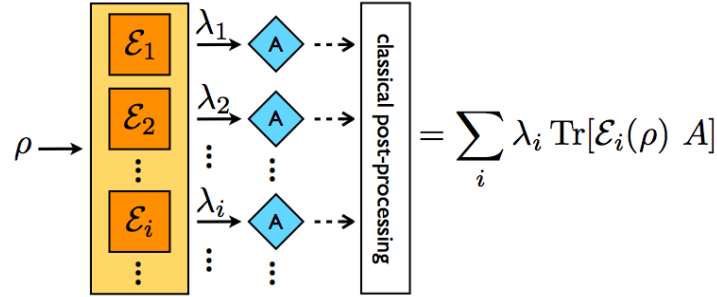


Fig. 1. Statistical decomposition of a non-physical transformation: (1) the initial state ρ goes through a quantum instrument, described by a collection of CP maps $\{\mathcal{E}_i\}_i$; (2) the outcome i , occurring with probability $p(i) = \text{Tr}[\mathcal{E}_i(\rho)]$, is recorded; (3) the corresponding output state $\rho_i = \mathcal{E}_i(\rho)/p(i)$ is used to evaluate the expectation value $\langle A \rangle_i = \text{Tr}[\rho_i A]$; (4) all data are finally recombined as $\sum_i \lambda_i p(i) \langle A \rangle_i = \sum_i \lambda_i \text{Tr}[\mathcal{E}_i(\rho) A]$, for suitable real coefficients λ_i .

4. Partial Expectation Values

The following representation theorem provides a realization scheme to access quantum correlation functions as in Eq. (1), in terms of *partial expectation values*.

Proposition 1 (Partial Expectation Values). *For any linear HP map $\mathcal{L} : \mathbf{L}(\mathcal{H}) \rightarrow \mathbf{L}(\mathcal{K})$, there exists a finite dimensional ancillary quantum system \mathcal{K}' , an isometry $V : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{K}'$ and an observable $Z \in \mathbf{L}(\mathcal{K}')$, such that*

$$\text{Tr}[V\rho V^\dagger (A \otimes Z)] = \text{Tr}[\mathcal{L}(\rho) A], \quad (2)$$

for all states $\rho \in \mathbf{S}(\mathcal{H})$ and all observables $A \in \mathbf{L}(\mathcal{K})$. Equivalently,

$$\mathcal{L}(\rho) = \text{Tr}_{\mathcal{K}'}[V\rho V^\dagger (\mathbf{1} \otimes Z)], \quad (3)$$

namely, the action of \mathcal{L} can be written as a partial expectation value.

Proof. Let $\mathcal{L}(\rho) = \sum_i \lambda_i \mathcal{E}_i(\rho)$ be a statistical decomposition of \mathcal{L} . Then, following Stinespring-Kraus's representation theorem ⁷, there exist \mathcal{K}' ancillary Hilbert space, $V : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{K}'$ isometry, and $\{P^i\}_i$ POVM on \mathcal{K}' such that

$$\mathcal{E}_i(\rho) = \text{Tr}_{\mathcal{K}'}[V\rho V^\dagger (\mathbf{1}_{\mathcal{K}} \otimes P_{\mathcal{K}'}^i)].$$

The statement is recovered by setting $Z := \sum_i \lambda_i P^i$. \square

Notice that Proposition 1 can be regarded as a generalization of Stinespring-Kraus's representation theorem ⁷ to arbitrary HP map. The idea of partial expectation values is depicted in Fig. 2 below.

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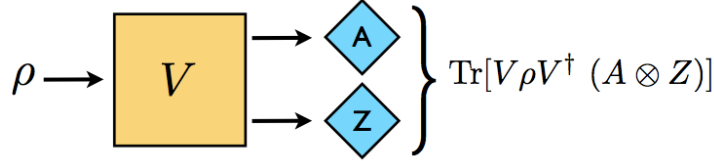


Fig. 2. Universal optimal strategy to access quantum correlation functions in terms of *partial expectation values*, see Proposition 1. The isometry V and the ancillary observable Z do not depend neither on the input state ρ nor on the final observable A , but only on the linear HP map \mathcal{L} . It holds that $\text{Tr}[V\rho V^\dagger (A \otimes Z)] = \text{Tr}[\mathcal{L}(\rho) A]$, for all input states ρ and all final observables A .

5. Conclusion

We provided a simple realization scheme for accessing quantum correlations, in terms of a unitary interaction U with an ancillary system followed by the measurement of an observable Z on the ancilla. Our scheme is universal, U and Z being independent of ρ , A , and B , and optimal, minimizing the statistical error associated with the classical postprocessing. Our scheme is suitable for implementation with present quantum optical technology, and represents a new way to test uncertainty relations⁵.

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